Initial Value Problems

Selim Kaan Ozsoy, Middle East Technical University

Definition. First-order differential equations are differential equations that only involve the first order derivatives of a specific variable with respect to another variable.

Examples:

$$\frac{dy}{dx} = 2x, \qquad y' = \arctan(x) + e^x, \qquad (2xe^x)\left(\frac{dy}{dx}\right) = \sin(\arctan(x))$$

Definition. Degrees of freedom refers to the number of independent data that can successfully estimate a parameter.

This may feel a little vague. But think about this; you have computed the result of an indefinite integral, say

$$\int 2x \; dx$$

and you get $x^2 + C$, where C is an arbitrary constant. How many predetermined data, or observation, do you need to determine the value of C? The answer is one. Assume that you are given the fact that when you plug 2 into x, you get the result 4. Then, you say, with a little abuse of notation,

$$(x = 2)^2 + C = 4 \tag{1}$$

$$4 + C = 4 \tag{2}$$

$$C = 0 \tag{3}$$

Therefore, we obtain the value of C, which means that one independent data is adequate to obtain the full solution, and so the degree of freedom is one.

In first-order differential equations, there only appears the first derivative of a variable, so, in the end, the degree of freedom of the differential equation is one. In general, it can be underlined that the degree of freedom of the solution of a differential equation is the same as its order.

It is better to start with an example.

$$y' = 2x^2y, \qquad y(0) = 1.$$

Here, we know that $y' = 2x^2y$ is an ordinary differential equation, furthermore, it is a first-order equation, whose degree of freedom is one. $y(0) = 1$ is called the initial condition. Together, it is called an Initial Value Problem.

These type of problems ask us to solve the differential equation first, and then, plugging in the values $x = 0$ and $y = 1$ simultaneously, to compute the value of the constant, and finally, have the solution of the initial value problem.

Notation: IVP stands for Initial Value Problem and it is used in general.

Example(1) Solve the IVP given by

$$y' = 2x^2y, \qquad y(0) = 1$$

Solution: We first solve the equation and obtain a general solution as follows:

$$\begin{aligned} \frac{dy}{dx} &= 2x^2y \\ \frac{1}{y}dy &= 2x^2dx \\ \int \frac{1}{y}\ dy &= \int 2x^2\ dx \\ \ln|y| &= \frac{2x^3}{3} + C \\ |y| &= \tilde{C} \cdot e^{\left(\frac{2x^3}{3}\right)} \end{aligned}$$

We emphasize the transformation from e^C to $\tilde{C}$, since both are arbitrary constants. Now, we have obtained a general solution. It follows from plugging in the initial condition that

$$\begin{aligned} |1| &= \tilde{C} \cdot e^0 \\ 1 &= \tilde{C} \cdot 1 \\ 1 &= \tilde{C} \end{aligned}$$

Therefore, the solution of the IVP is given by

$$|y| = \tilde{C} \cdot e^{\frac{2x^3}{3}}$$

Example(2) Solve the IVP given by

$$y' = \frac{(1+y^2)(1+x^2)}{xy}, \qquad y(1) = 1, \qquad x > 0$$

Solution: We can straightforwardly observe that we are also given the condition $x > 0$. It is common to assume that, therefore, we may end up with an expression where $\ln(x)$ appears. However, this observation does not affect our method to solve the problem. Similarly, we first solve the equation and obtain a general solution, and then, by plugging in the initial condition, get the solution of the IVP.

$$\frac{dy}{dx} = \frac{(1+y^2)(1+x^2)}{xy}$$
$$\left(\frac{y}{1+y^2}\right) dy = \left(\frac{1}{x} + x\right) dx$$
$$\int \left(\frac{y}{1+y^2}\right)\, dy = \int \left(\frac{1}{x} + x\right)\, dx$$
$$\frac{\ln(1+y^2)}{2} = \ln(x) + \frac{x^2}{2} + C$$

Now, we have the general solution. We, then, consider the initial condition, by plugging in $y = 1$ and $x = 1$:

$$\frac{\ln(1+1)}{2} = \ln(1) + \frac{1^2}{2} + C$$
$$\frac{\ln(2)}{2} = \frac{1}{2} + C$$
$$\frac{\ln(2) - 1}{2} = C$$
$$\ln\left(\sqrt{\frac{2}{e}}\right) = C$$

Therefore, we conclude that the solution of the IVP is given by the implicit equation

$$\frac{\ln\left(1+y^2\right)}{2} = \ln(x) + \frac{x^2}{2} + \ln\left(\sqrt{\frac{2}{e}}\right).$$