A Friendly Introduction to Differential Equations

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Introduction

The key to understanding differential equations is to first understand the meaning of differentials in the first place. Consider dx or dy, do they have a considerable meaning alone? The answer to this question is both yes and no, and this is actually one of the concepts that mathematicians and engineers, or applied mathematicians prefer dealing in different ways. In prerequisite calculus, the reader studied differentiation and is familiar with the expression $\frac{dy}{dx}$. This mathematical expression tells us how fast the dependent variable y changes.

In the study of differential equations, the symbols dx and dy will be treated as if they were fractions rather than operators, and we will assume that basic algebraic manipulations apply to these expressions.

It should be understood, however, that each such manipulation is ultimately justified by fundamental results of calculus, such as the chain rule or the manipulations of the limit definition of derivatives. Here, for simplicity, these justifications will not be proved, and will rely instead on intuitive reasoning. From now on, under these circumstances, we call dx, where x is the expression of any variable, a differential.

Now, let's talk about the equations that involve differentials, which are evidently called differential equations. Consider

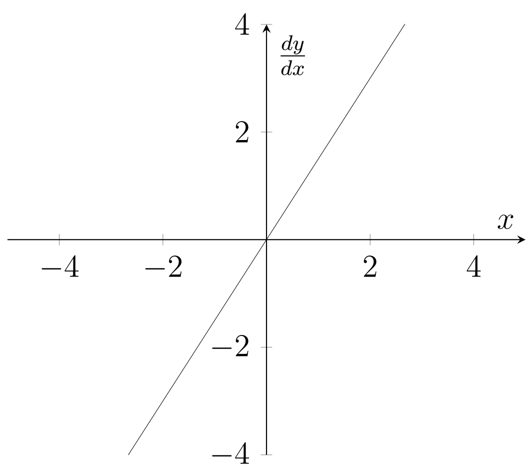
$$\frac{dy}{dx} = 2$$

This is a familiar expression. We infer, here, letting y be the dependent variable, that an infinitesimal change in x, say dx, results in a change of $2dx$ in y. But also, we can infer that the derivative of such y is equal to 2 in the corresponding domain, everywhere. Considering these, we can furthermore evolve our understanding and say y is a function of x and it is of the form $y(x) = 2x + C$, where C is an arbitrary constant. Apparently, C can take any value, so long as it is a constant, since the derivative of C with respect to x is identically zero.

But now, let's go beyond the usual. Consider:

$$\frac{dy}{dx} = 2x$$

Here, the rate of change of the variable y is given by an expression of another variable, x. That is, the behaviour of y at each point depends explicitly on the value of x at that point.



Considering these, the reader is encouraged to explore direction fields.