Separable Equations

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Now, we consider some specific methods to solve differential equations. The aim is mostly to consider y as a variable dependent on x and solve the equation for y. It is, however, common to leave the expressions in implicit forms. One of the most elementary types of differential equations is separable equations.

A first-order differential equation is said to be separable if it can be written in a form where all terms involving the dependent variable y and its differential dy appear on one side of the equation, while all terms involving the independent variable x and its differential dx appear on the other side. It looks like the following:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

or, in a better way,

$$g(y)dy = f(x)dx$$

Here, we construct our equation in such a form so that we can integrate easily. It is trivial to apply:

$$\int g(y)\,dy = \int f(x)\,dx$$

Letting, by Fundamental Theorem of Calculus, F be an anti-derivative of f, and G be an anti-derivative of g, we obtain:

$$G(y) + C_1 = F(x) + C_2.$$

Of course, C_1 and C_2 are arbitrary constants. Therefore, we can simply collect the constants in one side and say:

$$G(y) = F(x) + C.$$

Example (1) Solve the following differential equation for y explicitly.

$$\frac{dy}{dx} = 3y^2x^2$$

Solution: We wish to, firstly, collect the terms involving dy and y, and, dx and x in different sides. Here, we obtain

$$\frac{dy}{3y^2} = x^2dx.$$

The rest is trivial integration.

$$\int \frac{1}{3y^2}\,dy = \int x^2\,dx \tag{1}$$

$$-\frac{1}{3y} + C_1 = \frac{x^3}{3} + C_2 \tag{2}$$

$$-\frac{1}{3y} = \frac{x^3}{3} + C \tag{3}$$

$$y = y(x) = -\frac{1}{x^3 + \widetilde{C}} \tag{4}$$

Example (2) Solve the following differential equation for y explicitly

$$y' = 7(x+6)(y^2+1)$$

Solution: Now, observe that, in the equation in the example, there exists no explicit differentials. We, however, know that y' can be also expressed as $\frac{dy}{dx}$. Here, we note that, as a matter of fact, different notations of derivatives have historical meanings, but, in the course of differential equations, it makes it much easier to use Leibniz notation, that is, $\frac{dy}{dx}$. If the reader wants further information about the topic, y' is referred as Lagrange's notation, and $\dot{y}$ is referred as Newton's notation (of derivatives). Now, the motivation to solve the equation stays the same. We have:

$$\frac{dy}{dx} = 7(x+6)(y^2+1)$$

which is followed by

$$\frac{dy}{y^2+1} = 7(x+6)dx$$

Then, we may apply integration as follows,

$$\int \frac{1}{y^2+1} dy = \int 7(x+6)dx$$
$$\arctan(y) = \frac{7x^2}{2} + 42x + C$$

Here, we note that it is natural to leave $\arctan(y)$, since the transformation from $\arctan(y)$ to y requires a careful analysis of the interval of y, as the inverse trigonometric functions are defined on restricted intervals.