Strong Induction

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Theorem.(Strong Induction) Let $\phi(n)$ be a property of natural numbers. Suppose that $\phi(1)$ holds and for all $n \in \mathbb{N}$, if $\phi(1)$, $\phi(2)$, ..., $\phi(n)$ hold, then $\phi(n+1)$ holds. Then $\phi(n)$ holds for all $n \in \mathbb{N}$.

Also, we can express this theorem in a similar way to what we did in the previous note as follows:

Let $S \subseteq \mathbb{N}$. If

1. $1 \in S$
2. If $k \in S$, and $\{1, 2, 3, \ldots, k\} \subseteq S$, then $k + 1 \in S$

then, $S = \mathbb{N}$.

Example: Given integer $n \geq 0$, define a_n recursively as follows:

$$
\begin{aligned}
a_0 &= 1, \\
a_1 &= 3, \\
a_n &= 2a_{n-1} - a_{n-2} \quad \text{for } n \geq 2
\end{aligned}
$$

Show that for all $n \geq 0$, $a_n = 2n + 1$.

Solution:

Firstly, it should be noted that in this problem, $\mathbb{N}$ is considered as $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, unlike our previous problems or the lecture notes on number theory in general.

We will solve the problem by Strong Mathematical Induction.

Firstly, we can straightforwardly see that the statement is true, since

$$\begin{aligned}
&a_0 = 1 = 2 \cdot 0 + 1 \\
&a_1 = 3 = 2 \cdot 1 + 1 \\
&a_2 = 2 \cdot a_1 - a_0 = 5 = 2 \cdot 2 + 1 \\
&a_3 = 2 \cdot a_2 - a_1 = 7 = 2 \cdot 3 + 1 \\
&\vdots \\
&a_n = 2n + 1
\end{aligned}$$

Proof by Induction(Strong Induction):

Induction Basis:

For $n = 0$, we can observe that $a_0 = 1 = 2 \cdot 0 + 1$. So, for the base case, the statement is correct.

Inductive Step:

Assume, for a fixed but an arbitrary natural number $k \geq 0$ such that $0 \leq i \leq k$, that $a_i = 2i + 1$(Induction Hypothesis/Assumption).

We will show that $a_{k+1} = 2(k+1) + 1$.

Here, it should be noticed that, since $a_i = 2i + 1$ is satisfied when $i = 0$ and $i = 1$ and furthermore $i = 2$, we can assume that $k \geq 2$. It follows from the definition of a_n that $a_k = 2a_{k-1} - a_{k-2}$. But then, by induction hypothesis, since $k \geq 2 \geq 0$ and thus $k - 1 \geq 1 \geq 0$, we obtain

$$a_{k+1} = 2a_k - a_{k-1} = 2(2(k) + 1) - [2(k-1) + 1],$$

and so

$$a_k = (4k + 2) - (2k - 1),$$

which is followed by

$$a_{k+1} = 2k + 3 = 2(k+1) + 1.$$

Hence, by principle of strong induction, for all $n \geq 0$, $a_n = 2n + 1$.