Well Ordering Principle

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Well Ordering Principle: Every nonempty subset of $\mathbb{N}$ has a least element, i.e., there exists $a \in S$ such that $x \geq a$ for all $x \in S$.

This principle may seem a little obvious. The conditions, however, are considerable. Notice that, firstly, we need a nonempty set. The reason is trivial to comprehend, as the empty set has no elements and therefore it cannot have a least element. Furthermore, the set must be a subset of $\mathbb{N}$.

We can consider some sets and determine whether they satisfy the Well Ordering Principle, i.e., they are well ordered.

Example: Consider the set

$$S = \{x \in \mathbb{N} \mid x > 4\}.$$

Here, from the set notation, we conclude that S is a subset of $\mathbb{N}$. Also, we can show that the set is nonempty, as $10 \in \mathbb{N}$ and $10 > 4$ and therefore $10 \in S$. Hence, the set S satisfies the conditions of Well Ordering Principle and therefore S is well ordered. Moreover, the least element of S is 5. Since $5 \in S$ and for all $x \in S$, $x \geq 5$.

Example: Consider the set

$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

We can easily see that, letting $S = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, S is not a subset of $\mathbb{N}$. We can show this by the fact that $-3 \in S$ and $-3 \notin \mathbb{N}$ and therefore the exists some $x \in S$ such that $x \notin \mathbb{N}$, which implies that S is not a subset of $\mathbb{N}$, by definition of subsets. However, we cannot deduce that S is not well ordered just because S does not satisfy a condition of Well Ordering Principle. This is a common mistake in mathematical reasoning. Well Ordering Principle says that if a set satisfies some conditions, then it is guaranteed that it has a least element and thus it is a well ordered set, but it does not assert that if a set does not satisfy at least one condition, then it cannot be well ordered. Here, the reader may say "But $\{\dots,-3,-2,-1,0,1,2,3,\dots\}$ does not seem well ordered to me." and one could be right. In fact, $\{\dots,-3,-2,-1,0,1,2,3,\dots\}$ is not well ordered. Then how can we prove it?

Well Ordering Proofs:

Well Ordering proofs mainly consist of two types of proofs. In the first one, we show that a set is well ordered by showing the corresponding set satisfies the conditions of Well Ordering Principle. In the latter one, we show that the set is not well ordered by a contradiction, i.e., we first assume that the set is well ordered, and then obtain a contradiction from such an assumption. Example: Show that

$$S = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$$

is not well ordered.

Solution:

Assume towards a contradiction that S is a well ordered set. Then S has a least element. Let x be the least element of S. Then, we get $x \in S$ and for all $a \in S$, $a \geq x$. Here, it is clear that $S = \mathbb{Z}$ and since $x \in S$, $x - 1 \in S$, as both x and $x - 1$ are integers. But then, $x - 1 < x$, which is a contradiction to the fact that "for all $a \in S$, $a \geq x$". Therefore, by proof by contradiction, S is not well ordered.

Definition. Theorem is a statement which is true and can be proven.

Theorem. *If $a, b \in \mathbb{N}$, then there exists $n \in \mathbb{N}$ such that $na \geq b$.*

This theorem is also called the Archimedean Property of Integers.

Proof. Suppose towards a contradiction that the theorem is false and therefore there exist some natural numbers a, b such that $na < b$ for all $n \in \mathbb{N}$.

Let S be the set given by

$$S = \{b - na \mid n \in \mathbb{N}, b - na > 0\}.$$

Observe that S is a subset of $\mathbb{N}$, $b - na > 0$, since $b, n, a \in \mathbb{N}$ and so $b - na \in \mathbb{N}$. Furthermore, assuming that $n = 1 \in \mathbb{N}$ and $b > a$ and so $b - a > 0$, we have $b - a \in S$. Therefore, S is nonempty. Hence, according to Well Ordering Principle, S has a least element. Let x be the least element of S. Hence, by definition of S, $x = b - \tilde{n}a$ for some arbitrary $\tilde{n} \in \mathbb{N}$. But then, $b - (\tilde{n} + 1)a$ is also an element of S. It follows that $b - (\tilde{n} + 1)a < b - \tilde{n}a$, which contradicts the fact that $b - \tilde{n}a$ is the least element of S. $\square$